Thermodynamics in multiply connected spaces

1979 J. Phys. A: Math. Gen. 12 L309

(http://iopscience.iop.org/0305-4470/12/11/005)

J. Phys. A: Math. Gen., Vol. 12, No. 11, 1979. Printed in Great Britain

LETTER TO THE EDITOR

Thermodynamics in multiply connected spaces

S D Unwin†

Department of Theoretical Physics, The University cf Manchester, Manchester M13 9PL, England

Received 31 July 1979

Abstract. We perform a covariant calculation of the finite-temperature stress–energy–momentum matrix elements for the conformally coupled, massive scalar field in two flat, spatially compact, multiply connected spaces, $\mathbb{R}^1 \otimes G_2$ and $\mathbb{R}^1 \otimes B_1$, the latter being non-orientable. The interrelationship of the resulting elements is investigated from a thermodynamic viewpoint, and we find this interrelation to be independent of the conformal coupling chosen. Also, we consider the spin-1 field in the same spaces, and finally discuss thermodynamics in Robertson–Walker spacetimes with underlying manifolds $\mathbb{R}^1 \otimes G_2$ and $\mathbb{R}^1 \otimes B_1$.

1. Introduction

We shall first set about calculating the finite-temperature stress–energy–momentum matrix elements (which we may call the stress expectation values since the in and out vacua are the same) by covariant means (Dowker and Critchley 1976, Dowker and Banach 1978, De Witt *et al* 1978) for a conformally coupled, massive scalar field in two flat, spatially compact, multiply connected spaces, $\mathbb{R}^1 \otimes G_2$ and $\mathbb{R}^1 \otimes B_1$ (Wolf 1967). The interrelationship of the resulting elements will be investigated from a thermodynamic viewpoint, and we find these interrelations to be independent of the conformal coupling chosen. Also, we consider the spin-1 field in the same spaces, and finally discuss thermodynamics in curved spacetimes with the above underlying manifolds.

$\langle T_{\mu\nu} \rangle$ is calculated from the scalar field Feynman propagator $\Delta(x, x')$ for the spacetime satisfying

$$(\nabla_\mu \nabla^\mu + m^2 + R/6)\Delta(x, x') = -\delta(x, x').$$

Both multiply connected spaces are covered by $\mathbb{R}^4$, and choosing $g_{\mu\nu} = \operatorname{diag}(1, -1, -1, -1)$, we may put $R = 0$, $\nabla_\mu \to \partial_\mu$. $\Delta(x, x')$ is the propagator in $\mathbb{R}^1 \otimes \mathbb{R}^3/\Gamma$, Γ being isomorphic to the fundamental group of the multiply connected space, and is related to the propagator $\tilde{\Delta}(x, x')$ in Minkowski space by

$$\Delta(x, x') = \sum_\gamma \tilde{\Delta}(x, x'\gamma), \qquad \gamma \in \Gamma.$$

To obtain the finite-temperature propagator, we follow the prescription of Brown and

† Work supported by an SRC Research Studentship.

Maclay (1969), whereby

$$\Delta^{\beta}(x, x') = \sum_{r=-\infty}^{\infty} \Delta(x, x' - \mathrm{i}r\beta\lambda),$$

β being the inverse temperature and λ the timelike unit vector $(1, 0, 0, 0)$. We have taken the chemical potential to be zero. The orientable and non-orientable spaces, G_2 and B_1 respectively, can be represented by the point identifications

$$G_2\colon (x, y, z)\gamma \to [x + nl_1, (-1)^n y + 2pl_2, (-1)^n z + 2kl_3], \tag{1a}$$

$$B_1\colon (x, y, z)\gamma' \to [x + nl_1, (-1)^n y + 2pl_2, z + 2kl_3] \tag{1b}$$

in the covering space, where n, p, k are integers. Hence, our propagators are

$$\Delta^{\beta}_{G_2}(x, x') = \frac{m}{8\pi} \sum_{\substack{n,p,k,r \\ -\infty}}^{\infty} [(t - t' - \mathrm{i}r\beta)^2 - (\boldsymbol{x} - \boldsymbol{x}'\gamma_{npk})^2 - \mathrm{i}\epsilon]^{-1/2}$$

$$\times \mathrm{H}_1^{(2)}(m[(t - t' - \mathrm{i}r\beta)^2 - (\boldsymbol{x} - \boldsymbol{x}'\gamma_{npk})^2 - \mathrm{i}\epsilon]^{1/2}),$$

$\mathrm{H}_1^{(2)}$ being a Hankel function, and likewise for $\Delta^{\beta}_{B_1}(x, x')$ with γ replaced by γ'. We note that in each case the contribution from n-even terms gives the propagator for the spacetime with a 3-torus spatial section $(2l_1 \times 2l_2 \times 2l_3)$, the n-odd terms giving the G_2 and B_1 correction contributions. We shall likewise split the stress expectation values into the 3-torus value $\langle T_{\mu\nu}\rangle'$ and the G_2, B_1 corrections $\langle T_{\mu\nu}\rangle''_{G_2}$, $\langle T_{\mu\nu}\rangle''_{B_1}$ respectively. $\langle T_{\mu\nu}\rangle$ is expressed as the coincidence limit of a bilinear operator acting on the Feynman propagator (De Witt 1975, Dowker and Critchley 1976):

$$\langle T_{\mu\nu}\rangle = \mathrm{i} \lim_{x' \to x} [\tfrac{2}{3}\partial_{\mu}\partial_{\nu'} - \tfrac{1}{6}g_{\mu\nu'}g^{\lambda\sigma'}\partial_{\lambda}\partial_{\sigma'} - \tfrac{1}{6}(g_{\mu\rho'}\partial^{\rho'}\partial_{\nu'} + g_{\nu'\sigma}\partial^{\sigma}\partial_{\mu})$$

$$+ \tfrac{1}{6}g_{\mu\nu'}(\partial_{\rho}\partial^{\rho} + \partial_{\rho'}\partial^{\rho'} + 2m^2) + \tfrac{1}{6}g_{\mu\nu'}m^2]\Delta^{\beta}(x, x').$$

The properties of the propagators allow us to write

$$\lim_{x' \to x} \partial_{\mu'}\partial_{\nu'}\Delta^{\beta}(x, x') = \lim_{x' \to x} \delta_{\mu'}{}^{\mu}\delta_{\nu'}{}^{\nu}\partial_{\mu}\partial_{\nu}\Delta^{\beta}(x, x'),$$

and we obtain for the non-zero stress expectation values

$$\langle T_{\mu}{}^{\nu}\rangle' = \frac{-m^2}{4\pi^2} \sum_{\substack{n,p,k,r \\ -\infty}}^{\infty}{}' [\rho^{-2}\mathrm{K}_2(m\rho)\,\mathrm{diag}(1, 1, 1, 1)$$

$$- m\rho^{-3}\mathrm{K}_3(m\rho)\,\mathrm{diag}(r^2\beta^2, 4n^2l_1^2, 4p^2l_2^2, 4k^2l_3^2)],$$

$$\langle T_{\mu}{}^{\nu}\rangle''_{G_2} = \frac{-m^2}{12\pi^2} \sum_{\substack{n,p,k,r \\ -\infty}}^{\infty} \{\nu^{-2}\mathrm{K}_2(m\nu)\,\mathrm{diag}(5, 5, 1, 1) - m\nu^{-3}\mathrm{K}_3(m\nu)[3r^2\beta^2\,\mathrm{diag}(1, 0, 0, 0)$$

$$+ 3(2n + 1)^2 l_1^2\,\mathrm{diag}(0, 1, 0, 0) + 4(y - pl_2)^2\,\mathrm{diag}(1, 1, 0, 1)$$

$$+ 4(z - kl_3)^2\,\mathrm{diag}(1, 1, 1, 0)]\},$$

$$\langle T_2{}^3\rangle''_{G_2} = \frac{-m^3}{3\pi^2} \sum_{\substack{n,p,k,r \\ -\infty}}^{\infty} (y - pl_2)(z - kl_3)\nu^{-3}\mathrm{K}_3(m\nu),$$

$$\langle T_\mu{}^\nu\rangle''_{B_1} = \frac{-m^2}{12\pi^2} \sum_{\substack{n,p,k,r \\ -\infty}}^{\infty} [\![4\psi^{-2}\mathrm{K}_2(m\psi)\,\mathrm{diag}(1,1,0,1)$$

$$- m\psi^{-3}\mathrm{K}_3(m\psi)\{3\,\mathrm{diag}[r^2\beta^2, (2n+1)^2 l_1^2, 0, 4k^2 l_3^2]$$

$$+ 4(y - pl_2)^2\,\mathrm{diag}(1,1,0,1)\}]\!],$$

where

$$\rho \equiv (r^2\beta^2 + 4n^2 l_1^2 + 4p^2 l_2^2 + 4k^2 l_3^2)^{1/2},$$

$$\nu \equiv [r^2\beta^2 + (2n+1)^2 l_1^2 + 4(y - pl_2)^2 + 4(z - kl_3)^2]^{1/2},$$

$$\psi \equiv [r^2\beta^2 + (2n+1)^2 l_1^2 + 4(y - pl_2)^2 + 4k^2 l_3^2]^{1/2},$$

K_n is a modified Bessel function of the third kind, and in Σ', we drop the infinite $n = p = k = r = 0$ Minkowski space term. Figures 1 and 2 show plots of $\langle T_0{}^0\rangle' + \langle T_0{}^0\rangle''_{B_1}$ against y and β for certain values of the other parameters.

2. Thermodynamics

Components of the stress tensor expectation values are usually taken to describe the energy density, momentum and stresses of the quanta associated with the field. In the derived expressions for $\langle T_\mu{}^\nu\rangle$, the $r = 0$, zero-temperature contributions are the well-known Casimir or vacuum stresses, the physical interpretation of which is perhaps less clear, but may be associated with the concept of virtual particles. Here we shall show

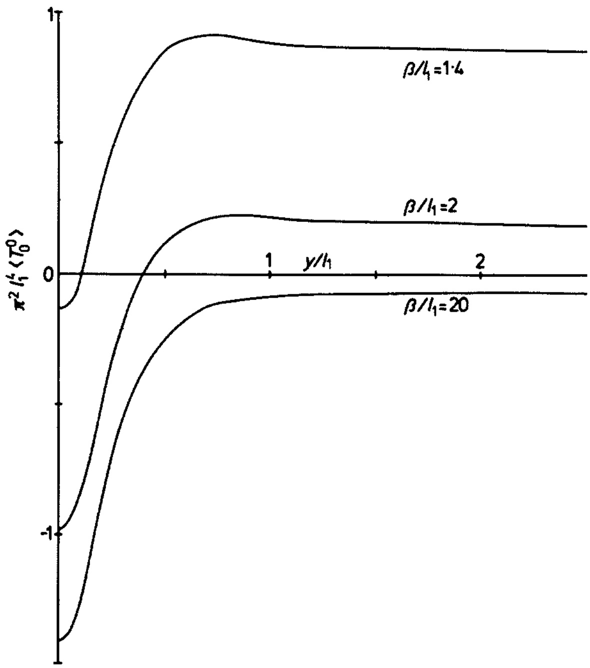

Figure 1. $\langle T_0{}^0\rangle' + \langle T_0{}^0\rangle''_{B_1}$ plotted against y/l_1 for l_2, $l_3 \to \infty$, $m = 0$. Symmetric about $y/l_1 = 0$.

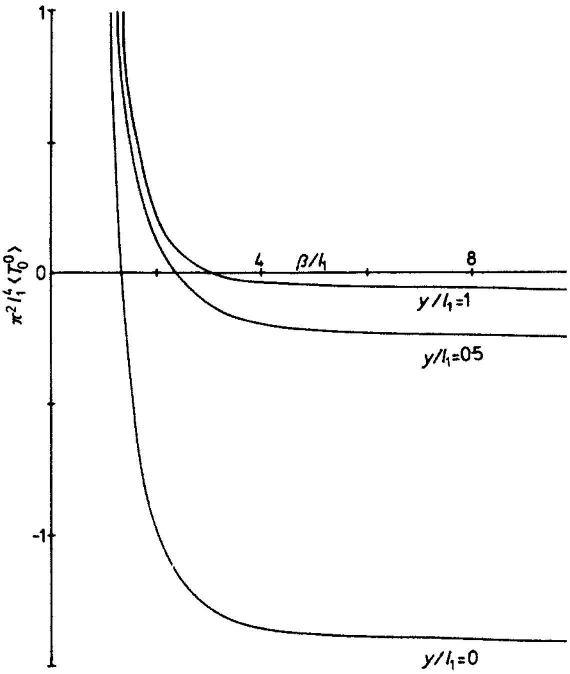

Figure 2. $\langle T_0{}^0\rangle' + \langle T_0{}^0\rangle''_{B_1}$ plotted against β/l_1 for $l_2, l_3 \to \infty$, $m = 0$.

that a simple thermodynamic relation exists between components of $\langle T_\mu{}^\nu\rangle$ for both the finite-temperature and vacuum parts.

The total energy E in either G_2 or B_1 is given by

$$E = \int_{V=4l_1l_2l_3} \langle T_0{}^0\rangle \, \mathrm{d}^3\boldsymbol{x}, \tag{2}$$

where $\langle T_0{}^0\rangle$ is the corresponding energy density $\langle T_0{}^0\rangle' + \langle T_0{}^0\rangle''$. We might expect the macroscopic properties of the quantum gas to satisfy

$$(\partial E/\partial V)_\beta = -[\partial(\beta P)/\partial\beta]_V, \tag{3}$$

although what we identify with the classical pressure P is not yet apparent. This equation is classically rooted in the principle of virtual work at a boundary, but in our case we require a different interpretation as we are considering spaces without boundaries. Looking at the 3-torus $\langle T_\mu{}^\nu\rangle'$, we see that the symmetry between inverse temperature and the torus dimensions manifests itself in equations which we may interpret to be the non-isotropic pressure equivalents to (3), that is

$$\partial(l_i\langle T_0{}^0\rangle')/\partial l_i = \partial(\beta\langle T_i{}^i\rangle')/\partial\beta \qquad \text{(no sum on } i\text{)}, \tag{4}$$

where we make the identifications $P_i' = -\langle T_i{}^i\rangle'$. The correction terms have a positional dependence, and we generalise (2) to define the quantities

$$\langle\widetilde{T_\mu{}^\nu}\rangle \equiv V^{-1}\int_V \langle T_\mu{}^\nu\rangle \, \mathrm{d}^3\boldsymbol{x}$$

such that $E = V\langle\widetilde{T_0{}^0}\rangle$. We see that $\langle T_\mu{}^\nu\rangle' = \langle T_\mu{}^\nu\rangle'$, and the non-zero components of $\langle\widetilde{T_\mu{}^\nu}\rangle''$ are

$$\langle\widetilde{T_\mu{}^\nu}\rangle''_{G_2} = \frac{-m}{8\pi l_2 l_3}\sum_{n,r \atop -\infty}^{\infty}\{\omega^{-1}\mathrm{K}_1(m\omega)\,\mathrm{diag}(1,1,0,0)$$

$$-m\omega^{-2}\mathrm{K}_2(m\omega)\,\mathrm{diag}[r^2\beta^2,(2n+1)^2 l_1^2,0,0]\},$$

$$\langle\widetilde{T_\mu{}^\nu}\rangle''_{B_1} = \frac{-m^2}{4\pi^2 l_2}\sqrt{\frac{\pi}{2m}}\sum_{n,r,k \atop -\infty}^{\infty}\{\tau^{-3/2}\mathrm{K}_{3/2}(m\tau)\,\mathrm{diag}(1,1,0,1)$$

$$-m\tau^{-5/2}\mathrm{K}_{5/2}(m\tau)\,\mathrm{diag}[r^2\beta^2,(2n+1)^2 l_1^2,0,4k^2 l_3^2]\},$$

where $\omega \equiv [r^2\beta^2+(2n+1)^2 l_1^2]^{1/2}$ and $\tau \equiv [r^2\beta^2+(2n+1)^2 l_1^2+4k^2 l_3^2]^{1/2}$. We now find

$$\partial(l_i\langle\widetilde{T_0{}^0}\rangle)/\partial l_i = \partial(\beta\langle\widetilde{T_i{}^i}\rangle)/\partial\beta \qquad \text{(no sum on } i\text{)}, \tag{5}$$

where $\langle T_\mu{}^\nu\rangle = \langle T_\mu{}^\nu\rangle' + \langle T_\mu{}^\nu\rangle''$.

3. The spin-1 field

The massive and massless spin-1 field stress expectation values may be expressed as multiples of the minimally coupled scalar field values, relating fields of the same mass in the same space (Unwin 1979). Since $\partial G_2 = \phi = \partial B_1$, the scalar $\langle\widetilde{T_\mu{}^\nu}\rangle$ are independent of the conformal coupling chosen (De Witt 1975), and we may give the relevant spin-1 values in terms of the previously calculated scalar values. We have, for the massless spin-1 field,

$${}_0\langle T_\mu{}^\nu\rangle' = 2\langle T_\mu{}^\nu\rangle', \qquad {}_0\langle\widetilde{T_\mu{}^\nu}\rangle''_{G_2} = -2\langle\widetilde{T_\mu{}^\nu}\rangle''_{G_2}, \qquad {}_0\langle\widetilde{T_\mu{}^\nu}\rangle''_{B_1} = 0,$$

and for the massive spin-1 field,

$${}_m\langle T_\mu{}^\nu\rangle' = 3\langle T_\mu{}^\nu\rangle', \qquad {}_m\langle\widetilde{T_\mu{}^\nu}\rangle''_{G_2} = -\langle\widetilde{T_\mu{}^\nu}\rangle''_{G_2}, \qquad {}_m\langle\widetilde{T_\mu{}^\nu}\rangle''_{B_1} = \langle\widetilde{T_\mu{}^\nu}\rangle''_{B_1}.$$

Hence the spin-1 $\langle\widetilde{T_\mu{}^\nu}\rangle$ values satisfy (5).

4. Discussion

Classically, the pressure may be defined in terms of the Helmholtz free energy F as

$$P_i = -l_i V^{-1}(\partial F/\partial l_i)_{\beta, l_{j\neq i}},$$

where

$$E = [\partial(\beta F)/\partial\beta]_V,$$

and these pressures then satisfy

$$l_i V^{-1}(\partial E/\partial l_i)_{\beta, l_{j\neq i}} = -[\partial(\beta P_i)/\partial\beta]_V,$$

the modification of (3) for non-isotropic pressure. These pressures coincide exactly with the previously calculated volume-averaged field pressures $-\langle\widetilde{T_i{}^i}\rangle$ (no sum), which may be shown explicitly using the expressions for the scalar field Helmholtz free energies $F = F' + F''$,

$$F' = \frac{-m^2 l_i l_2 l_3}{\pi^2} \sum_{\substack{n,p,k,r\\-\infty}}^{\infty}{}' \rho^{-2} \mathrm{K}_2(m\rho), \qquad F''_{G_2} = \frac{-ml_1}{2\pi} \sum_{\substack{n,r\\-\infty}}^{\infty} \omega^{-1} \mathrm{K}_1(m\omega),$$

$$F''_{B_1} = \frac{-m^2 l_1 l_3}{\pi^2} \sqrt{\frac{\pi}{2m}} \sum_{\substack{n,r,k\\-\infty}}^{\infty} \tau^{-3/2} \mathrm{K}_{3/2}(m\tau),$$

F' being half the 3-torus value. We conclude that the principle of virtual work applies over the volume as a whole rather than the idea of work being done at a boundary. Dowker and Critchley (1977) reach a similar conclusion for the case of a scalar field propagating in an Einstein universe S^3.

We note that the quantities $\langle \widetilde{T_\mu{}^\nu} \rangle$, and hence (5), are independent of the conformal coupling constant chosen for the scalar field. In the limit $m \to 0$, the scalar field considered is conformally invariant, as is the $m = 0$ spin-1 (electromagnetic) field, so in each case we have $\langle T_\mu{}^\mu \rangle = 0$, and hence $\langle \widetilde{T_\mu{}^\mu} \rangle = 0$, which may be rewritten

$$E = \sum_{i=1}^{3} P_i V. \tag{6}$$

We interpret this to be the equation of state for an ideal quantum gas when the pressure is non-isotropic. We see that conformal invariance of the massless scalar field is not required for (6) to be satisfied.

So far we have considered flat spacetimes, but of course, specifying the manifold, in our case $\boldsymbol{R}^1 \otimes G_2$ or $\boldsymbol{R}^1 \otimes B_1$, does not uniquely define the local spacetime structure. (See Geroch (1971) for an interesting discussion of general relativity from a global viewpoint.) The Robertson–Walker line element

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2), \tag{7}$$

for example, may describe the local structure of a spacetime with the underlying manifold $\boldsymbol{R}^1 \otimes G_2$ or $\boldsymbol{R}^1 \otimes B_1$, provided we make the appropriate point identifications ($1a$) or ($1b$) in the flat spatial sections. The concept of thermal equilibrium in non-stationary spacetimes has been discussed elsewhere, and we refer the reader to a Letter by Kennedy (1978) and the references therein. The metric (7) is conformally related to the flat metric, and the conformal invariance of the electromagnetic and massless scalar fields considered allows a straightforward calculation of $\langle T_\mu{}^\nu \rangle^{\mathrm{R-W}}$. In terms of the flat space results $\langle T_\mu{}^\nu \rangle$, we have

$$\langle T_\mu{}^\nu \rangle^{\mathrm{R-W}} = a^{-4}(t) \langle T_\mu{}^\nu \rangle + \langle \theta_\mu{}^\nu \rangle$$

for either field, where $\langle \theta_\mu{}^\nu \rangle$ is a function only of a and its time derivatives (Bunch and Davies 1977), and depends upon the field considered. The $a^{-4}(t)$ effectively scales the parameters, $l_i \to a(t) l_i$, $\beta \to a(t)\beta$ and $\boldsymbol{x} \to a(t)\boldsymbol{x}$ in $\langle T_\mu{}^\nu \rangle$, describing an expanding and cooling universe (if $\dot{a} > 0$), while $\langle \theta_\mu{}^\nu \rangle$ is the source of the familiar trace anomaly. We discover that the finite-temperature contribution to $\langle T_\mu{}^\nu \rangle^{\mathrm{R-W}}$ satisfies the thermodynamic relation (5), whereas the same is not true of the vacuum contribution. The latter may be separated into two distinct parts, depending upon the physical origins of the stress. Firstly, there is the zero-temperature part of $a^{-4}(t) \langle T_\mu{}^\nu \rangle$ which is a consequence of the closure of the manifold, that is, results from the global properties of the spacetime, and this part satisfies (5). The second part, $\langle\theta_\mu{}^\nu\rangle$ is a curvature effect, that is, a consequence of local properties of the spacetime, and does not satisfy (5).

References

Brown L S and Maclay G J 1969 *Phys. Rev.* **184** 1272
Bunch T S and Davies P C W 1977 *Proc. R. Soc.* A **356** 569
De Witt B S 1975 *Phys. Rep.* **19**C 295
De Witt B S, Hart C F and Isham C J 1978, *Preprint* University of Texas, Austin
Dowker J S and Banach R 1978 *J. Phys. A: Math. Gen.* **11** 2255
Dowker J S and Critchley R 1976 *J. Phys. A: Math. Gen.* **9** 535
——— 1977 *Phys. Rev.* D **15** 1484
Geroch R 1971 *Proc. Int. School of Physics 'Enrico Fermi' Course XLVII* (New York: Academic)
Kennedy G 1978 *J. Phys. A: Math. Gen.* **11** L77
Unwin S D 1979 *J. Phys. A: Math. Gen.* **12**
Wolf J 1967 *Spaces of Constant Curvature* (New York: McGraw-Hill)